§14.8

8. Extrema on a curve Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.

$$f(x, y) := x^2 + y^2$$

$$g(x, y) := x^2 + xy + y^2 - 1$$

$$\nabla f = (2x, 2y)$$

$$\nabla g = (2x + y, x + 2y)$$

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \begin{cases} 2x = 2\lambda x + \lambda y & -(1) \\ 2y = \lambda x + 2\lambda y & -(2) \end{cases}$$

$$(1) \Rightarrow \quad \lambda = \frac{2x}{2x + y}$$

$$(2) \Rightarrow \quad 2y = \frac{2x}{(2x + y)}(x + 2y)$$

$$\Rightarrow \quad 4xy + 2y^2 = 2x^2 + 4xy$$

$$\Rightarrow \quad x^2 = y^2$$

$$\Rightarrow \quad x = \pm y.$$

$$x = y,\ g(x, y) = 0 \Rightarrow \quad x^2 + x^2 + x^2 = 1$$

$$\Rightarrow \quad x = y = \pm \frac{1}{\sqrt{3}}$$

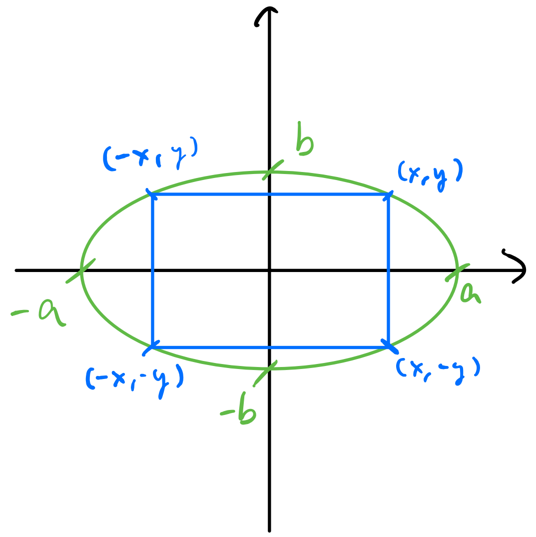
$$x = -y,\ g(x, y) = 0 \Rightarrow \quad x^2 - x^2 + x^2 = 1$$

$$\Rightarrow \quad x = \pm 1,\ y = \mp 1$$

$\therefore (1, -1)$ and $(-1, 1)$ are furthest, dist. $= \sqrt{2}$

$(1/\sqrt{3}, 1/\sqrt{3})$ and $(-1/\sqrt{3}, -1/\sqrt{3})$ are closest, dist. $= \sqrt{\frac{2}{3}}$.

12. Rectangle of longest perimeter in an ellipse Find the dimensions of the rectangle of largest perimeter that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ with sides parallel to the coordinate axes. What *is* the largest perimeter?



As illustrated in the graph,
a rectangle satisfying the conditions
(inscribed in that ellipse, sides parallel to the axes)
depends only on one point (x, y)
in the first quadrant,
with dimension $2x \times 2y$,
perimeter $2(2x + 2y)$

$f(x, y) := x + y$, $\nabla f = (1, 1)$

$g(x, y) := \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$, $\nabla g = \left(\frac{2}{a^2} x , \frac{2}{b^2} y \right)$

$\nabla f = \lambda \nabla g$

$\Rightarrow \quad x = \frac{a^2}{2\lambda}$, $y = \frac{b^2}{2\lambda} = \frac{b^2}{2 \cdot \left(\frac{a^2}{2x} \right)} = \frac{b^2}{a^2} x$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{b^4}{a^4} \cdot \frac{x^2}{b^2} = 1$

$\Rightarrow \frac{(a^2 + b^2)}{a^4} x^2 = 1$

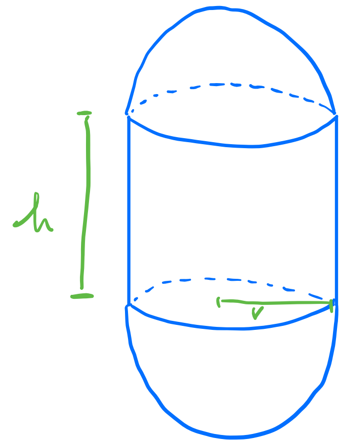
$\Rightarrow \quad x = \frac{a^2}{\sqrt{a^2 + b^2}}$ or $- \frac{a^2}{\sqrt{a^2+b^2}}$ (rej)

$\Rightarrow \quad y = \frac{b^2}{\sqrt{a^2 + b^2}}$

$\therefore$ height $= 2x = \frac{2a^2}{\sqrt{a^2+b^2}}$, perimeter $= 4x + 4y$

width $= 2y = \frac{2b^2}{\sqrt{a^2+b^2}}$, $\quad = 4\sqrt{a^2 + b^2}$

16. Cheapest storage tank Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a cylindrical tank with hemispherical ends, and the tank is to hold 8000 m^3 of gas. The customer also wants to use the smallest amount of material possible in building the tank. What radius and height do you recommend for the cylindrical portion of the tank?



The volume of material $\approx$ surface area $\times$ thickness

Consider only the surface area:

$f(r,h) := 2\pi r h + 4\pi r^2$

Capacity of the tank:

$g_1(r,h) := \pi r^2 h + \frac{4}{3}\pi r^3$

$g(r,h) := g_1(r,h) - 8000$

$\nabla f = (2\pi h + 8\pi r, 2\pi r)$

$\nabla g = (2\pi r h + 4\pi r^2, \pi r^2)$

$\nabla f = \lambda \nabla g$

$$\Rightarrow \begin{cases} 2\pi h + 8\pi r = \lambda(2\pi r h + 4\pi r^2) & \text{— (1)} \\ 2\pi r = \lambda \pi r^2 & \text{— (2)} \end{cases}$$

(2) $\Rightarrow$ $\lambda = 2/r$ or $r = 0$ (rej.)

(1) $\Rightarrow$ $2\pi h + 8\pi r = 4\pi h + 8\pi r$

$\Rightarrow$ $h = 0$

$\Rightarrow \frac{4}{3}\pi r^3 = 8000 \Rightarrow r^3 = \frac{6000}{\pi}$

$\Rightarrow r = 10\sqrt[3]{\frac{6}{\pi}}$.

$\therefore$ Optimal radius $= 10\sqrt[3]{\frac{6}{\pi}}$ m,

height $= 0$ m.

17. Minimum distance to a point Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$.

Let $(x', y', z') = (x-1, y-1, z-1)$

Then distance between (x, y, z) and $(1, 1, 1)$

= distance between (x', y', z') and origin.

$$f(x', y', z') := (x')^2 + (y')^2 + (z')^2$$

Constraint : $x + 2y + 3z = 13$

$\Rightarrow (x-1) + 2(y-1) + 3(z-1) = 7$

$\Rightarrow \quad x' + 2y' + 3z' = 7$

$$g(x', y', z') := x' + 2y' + 3z' - 7.$$

$$\nabla f = (2x', 2y', 2z')$$

$$\nabla g = (1, 2, 3).$$

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \begin{cases} x' = \frac{1}{2}\lambda \\ y' = \lambda \\ z' = \frac{3}{2}\lambda \end{cases}$$

$$g(x', y', z') = 0$$

$\Rightarrow \frac{1}{2}\lambda + 2\lambda + \frac{9}{2}\lambda = 7$

$\Rightarrow \quad \lambda = 1$

$\therefore (x', y', z') = (\frac{1}{2}, 1, \frac{3}{2})$

$\therefore (x, y, z) = (\frac{3}{2}, 2, \frac{5}{2})$

with minimal distance $(= \sqrt{\frac{7}{2}})$

24. Extrema on a sphere Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum values.

$f(x,y,z) = x + 2y + 3z$, $\nabla f = \langle 1, 2, 3 \rangle$.

$g(x,y,z) = x^2 + y^2 + z^2 - 25$, $\nabla g = \langle 2x, 2y, 2z \rangle$.

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \begin{cases} 1 = 2\lambda x \\ 2 = 2\lambda y \\ 3 = 2\lambda z \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = \frac{1}{\lambda} \\ z = \frac{3}{2\lambda} \end{cases}$$

$$g(x,y,z) = 0$$

$$\Rightarrow \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{9}{4\lambda^2} = 25$$

$$\Rightarrow \lambda^2 = \frac{14}{100}$$

$$\Rightarrow \lambda = \pm \frac{\sqrt{14}}{10}$$

$5 + 20 + 45$

$25 \quad 45$

$70.$

$$\Rightarrow (x,y,z) = \left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right)$$

$$\text{or } (x,y,z) = \left(-\frac{5}{\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{15}{\sqrt{14}}\right)$$

$\therefore$ Max: $\left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right)$ with $f = \frac{70}{\sqrt{14}} = 5\sqrt{14}$

Min: $\left(-\frac{5}{\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{15}{\sqrt{14}}\right)$ with $f = -\frac{70}{\sqrt{14}} = -5\sqrt{14}$

34. Blood types Human blood types are classified by three gene forms A, B, and O. Blood types AA, BB, and OO are *homozygous*, and blood types AB, AO, and BO are *heterozygous*. If p, q, and r represent the proportions of the three gene forms to the population, respectively, then the *Hardy-Weinberg Law* asserts that the proportion Q of heterozygous persons in any specific population is modeled by

$$Q(p, q, r) = 2(pq + pr + qr),$$

subject to $p + q + r = 1$. Find the maximum value of Q.

$$g(p, q, r) = p + q + r - 1$$

$$\nabla Q = (2(q+r), 2(p+r), 2(p+q))$$

$$\nabla g = (1, 1, 1)$$

$$\nabla Q = \lambda \nabla g$$

$$\Rightarrow \begin{cases} 2q + 2r = \lambda & -(1) \\ 2p + 2r = \lambda & -(2) \\ 2p + 2q = \lambda \end{cases}$$

$$(1), (2) \Rightarrow 2q + 2r = 2p + 2r$$

$$\Rightarrow q = p$$

Similarly, $p = q = r$

$$p + q + r = 1 \Rightarrow p = q = r = \frac{1}{3}.$$

$$\therefore \text{Max. } Q = 2\left(\frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3}\right)$$

$$= \frac{2}{3}$$

37. Maximize the function $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

$$\nabla f = (2x, 2, -2z).$$

$$g_1(x, y, z) := 2x - y \Rightarrow \nabla g_1 = (2, -1, 0)$$

$$g_2(x, y, z) := y + z \Rightarrow \nabla g_2 = (0, 1, 1).$$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$\Rightarrow \begin{cases} 2x = 2\lambda \\ 2 = -\lambda + \mu \\ -2z = \mu \end{cases}$$

$$\Rightarrow x = \lambda, \quad 2 = -x - 2z$$

$$\Rightarrow x = -2z - 2$$

$$2x - y = 0 \Rightarrow -4z - 4 - y = 0$$

$$\Rightarrow -3z - 4 - (y + z) = 0$$

$$\Rightarrow z = -\frac{4}{3}$$

$$\therefore (x, y, z) = \left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right).$$

is max. with value $= \frac{4}{3}$.

44. Minimum distance to the origin Find the point closest to the origin on the curve of intersection of the plane $2y + 4z = 5$ and the cone $z^2 = 4x^2 + 4y^2$.

$f(x,y,z) := x^2 + y^2 + z^2 \quad \nabla f = (2x, 2y, 2z)$

$g_1(x,y,z) := 2y + 4z - 5 \quad \nabla g_1 = (0, 2, 4)$

$g_2(x,y,z) := 4x^2 + 4y^2 - z^2 \quad \nabla g_2 = (8x, 8y, -2z)$

$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$

$$\Rightarrow \begin{cases} 2x = 8\mu x & \text{— (1)} \\ 2y = 2\lambda + 8\mu y & \text{— (2)} \\ 2z = 4\lambda - 2\mu z & \text{— (3)} \end{cases}$$

$(1) \Rightarrow \mu = \frac{1}{4}$ or $x = 0$

(Case 1) $x = 0 \Rightarrow z^2 = 4y^2$

$\Rightarrow z = 2y$ or $z = -2y$

$\Rightarrow (z) + 4z = 5$ or $(-z) + 4z = 5$

$\Rightarrow y = \frac{1}{2},\ z = 1$ or $y = \frac{5}{6},\ z = \frac{5}{3}$

(Case 2) $\mu = \frac{1}{4} \Rightarrow (2): 2y = 2\lambda + 2y$

$\Rightarrow \lambda = 0$

$\Rightarrow (3): 2z = -\frac{1}{2} z$

$\Rightarrow z = 0$

$g_2(x,y,z) = 0 \Rightarrow x = y = z = 0$

contradict to $g_1(x,y,z) = 0$

$f(0, \frac{1}{2}, 1) = \frac{5}{4}, \quad f(0, \frac{5}{6}, \frac{5}{3}) = \frac{125}{36}$

$\therefore (0, \frac{1}{2}, 1)$ is closest to origin with distance $= \frac{\sqrt{5}}{2}$.

Addition Questions:

(1) Consider system of equation

$$\begin{cases} 2x - y + z = 0 \\ e^{2x} + e^{-2y} + \sin z = 2 \end{cases}$$

which has a solution $(x,y,z) = (0,0,0)$.

Is (x,y) can be solved as functions of z, $x = x(z)$ & $y = y(z)$, near this point $(0,0,0)$?

If so, calculate the derivatives $\frac{dx}{dz}$, $\frac{dy}{dz}$ at the point.

$$F_1(x,y,z) := 2x - y + z$$

$$F_2(x,y,z) := e^{2x} + e^{-2y} + \sin z.$$

$$\frac{\partial F_1}{\partial x} = 2 \qquad \frac{\partial F_1}{\partial y} = -1 \qquad \frac{\partial F_1}{\partial z} = 1$$

$$\frac{\partial F_2}{\partial x} = 2e^{2x} \qquad \frac{\partial F_2}{\partial y} = -2e^{-2y} \qquad \frac{\partial F_2}{\partial z} = \cos z$$

$$\begin{vmatrix} \frac{\partial F_1}{\partial x}\Big|_{(0,0,0)} & \frac{\partial F_1}{\partial y}\Big|_{(0,0,0)} \\ \frac{\partial F_2}{\partial x}\Big|_{(0,0,0)} & \frac{\partial F_2}{\partial y}\Big|_{(0,0,0)} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} = -2 \neq 0$$

$\therefore$ Yes, by Implicit Function Thm

$$\begin{bmatrix} \frac{dx}{dz} \\ \frac{dy}{dz} \end{bmatrix} = - \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \cos 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -2 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$$

(2) Let $f(x,y) = \begin{pmatrix} x^3 - 3xy^2 \\ 3x^2y - y^3 \end{pmatrix}$

Show that for $(x,y) \neq (0,0)$, f has a local inverse.

$$Df(x,y) = \begin{bmatrix} 3x^2 - 3y^2 & -6xy \\ 6xy & 3x^2 - 3y^2 \end{bmatrix}$$

$$\det Df(x,y) = (3x^2 - 3y^2)^2 + 36x^2y^2$$

Note $(3x^2 - 3y^2)^2 \geq 0$ and $36x^2y^2 \geq 0$.

$\det Df(x,y) = 0 \iff (3x^2 - 3y^2)^2 = 0$
and $36x^2y^2 = 0$

For $(3x^2 - 3y^2)^2 = 0$,

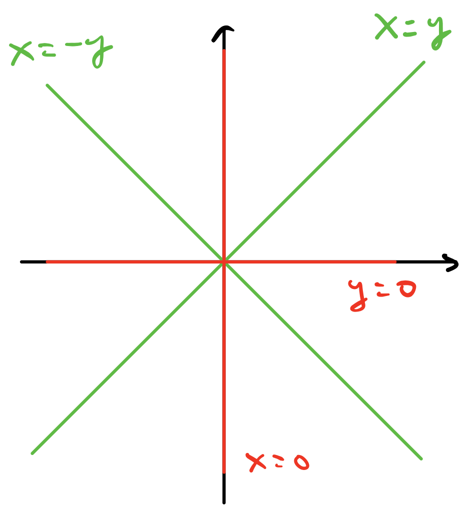
we need $x^2 = y^2$

i.e. $x = y$ or $x = -y$

For $36x^2y^2 = 0$,

we need $xy = 0$

i.e. $x = 0$ or $y = 0$



$\therefore$ $(0,0)$ is the only point satisfy $Df(x,y) = 0$

$\therefore$ For $(x,y) \neq (0,0)$, f has a local inverse,

by Inverse Function Theorem.